BRAINWAVES REPORT BW/003

TOWARDS A MATHEMATICAL REVOLUTION

The average person, if threatened with a knife, would regard the situation as serious but not totally hopeless: he might escape, or fend off the attacker, or pacify him. But consider the same person faced with a mortal danger he could only survive by giving the correct answer to the question: "How much is a tenth of one per cent divided by ten to the power of minus three?"

It would most likely be his last living experience, as he would not even attempt *to survive. A pity, for the complexity of understanding this task is less than that of understanding an average paragraph in a newspaper.*

(Ilan Samson, *Demathtifying: Demystifying Mathematics* (Chesham: QED Books, 2004), p.1.)

And here is where a more frightening thing is beginning to take effect: the loss among large numbers of people of the ability to think. In an era when our cars can tell us which way to turn, we are letting our computers do our thinking for us. How soon therefore before our cars tell us how to vote on the way to the polling station?

(Ruth Gledhill, 'Our new Dark Age', *Times 2*, 21 February 2001, p.3.)

The approaching century suggests a new beginning, and we can use this as an excuse to revitalize the general perception of mathematics. We must find teachers who truly love mathematics to teach our young. This places personal responsibility on each person who is enthralled by mathematics to contribute to improving primary and secondary mathematics education. We can have a general population that likes and respects mathematics, both in its elegant beauty and its usefulness.

(Calvin C. Clawson, *Mathematical Mysteries: The Beauty and Magic of Numbers* (New York: Basic Books, 1996), p.215.)

THE OPPORTUNITY

One of the most vital tasks for any government is the identification and execution of the most cost-effective attainable goals for the services and programmes under its control.

I wish to argue that the greatest single, affordable, benefit to our national life which it is in the remit of any government (as opposed to, say, the Church) to achieve would be to raise the mathematical consciousness of the nation. If this is an overstatement, it is an overstatement of a nevertheless valuable truth.

For we have been breeding successive generations to whom mathematics is dull, dreary, fearful, and indeed a closed book which is lacking in sense, direction and value. Small wonder that so many reject the subject at the first opportunity, never to take it up again. And the cost to our national life and economy is incalculable. Wherever we look, people are functioning at a level far below their natural potential. We are losing the ability to Think.

In reality mathematics offers - more than any discipline known to the West except perhaps the teaching of the classics, and particularly Latin - an education in the art of rational, creative *Thinking.* And I submit that there are few more desirable goals one could set before any country than that it become a nation of *Thinkers*.

Putting it another way, the most important goal of mathematical education is to teach people to Think by teaching them to think like a mathematician. This will surprise many who thought that the function of mathematics was to teach us to do calculations which can now be done far more easily, more quickly, and more accurately, by a calculator. But it would not have surprised Plato, who in *The Republic* prescribed mathematics as an essential feature in the education of the wise man. In the handling of abstract concepts and the insistence upon rigorous attention to detail the study of mathematics is without rival.

By the same token the practice of it inculcates economy of thought and tidiness of mind. As such it is foundational to Occam's Razor - the ancient principle that *entities are not to be multiplied beyond what is necessary* - which, besides being essential to scholarship is fundamental also to science. Einstein enshrined it when he declared: 'The grand aim of science [is] to cover the greatest number of empirical facts by logical deduction from the smallest number of hypotheses or axioms.'[1]

The human brain needs to be challenged and stimulated towards logical, creative reasoning, reasoning about the unfamiliar, in order to grow and stay healthy, just as the body needs exercise. Without such, both will atrophy.

Yet today except among a minority who work in the sciences, there is a pervading, almost universal terror of mathematics. The consequences of this are all around us, but in the main unrecognised. If each of us were trained in the arts of abstract thought that enabled us to handle the unfamiliar, we could solve many of the problems of today and collectively transform the life of this country.

I believe that this is a good idea whose time has come. And I call to witness the present day explosive success of the Su Doku phenomenon all over the country. People are rediscovering there the positive delight that comes from teaching themselves inductively to reason in unfamiliar ways. There is a vacuum here waiting to be filled.

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1 Albert Einstein, in Lincoln Barnett, *The Universe and Dr Einstein* (1950 edition).

THE ECONOMIC NEED: EFFICIENCY

It is a fact that one or two gifted mathematicians can service an entire company of several hundred people. There are few areas to which the trained mathematician cannot turn his or her skills . It is the mathematicians who can provide the theory to support our research. A few hours spent in mathematics can save hundreds of hours in computer time. And even those untalented in the subject will function better at almost any level of business if they are encouraged to grow in numeracy and habitually relish the stimulus of stretching their brains with unfamiliar challenges.

As Sir Adrian Smith's report put it, 'An adequate supply of young people with mastery of appropriate mathematical skills at all levels is vital to the future prosperity of the UK.'[2]

THE SOCIAL NEED: ABSOLUTES

All human society needs a basis of shared absolutes if it is to cohere. In today's Britain these are under threat. There is one respect in which mathematics can offer assistance.

For mathematics embodies in abundance universals whose very possibility much contemporary postmodern thinking denies. Two and two would have equalled four even had there been no big bang, and will continue to do so long after the universe has reached whatever end it finally comes to. If the constants e and π did not hold the values that they do, no universe would be imaginable. Any intelligent being on the far side of the universe will arrive (within limits of precision) at the same values for these constants as we have. Any theorem we can prove on earth will necessarily be true anywhere else in the universe. This is why mathematicians today with few exceptions ascribe a genuine objective reality to the subjects of their enquiries, just as Plato did in fourth century Athens.[3] Further, as follows from Euler's brilliant achievement in unifying the mathematics of his day in 1748, any wave motion and any particle anywhere necessarily embody the core mathematical concepts of exponentials, logarithms, trigonometry and complex numbers - just as, we are told, every cell in our bodies incorporates our DNA. That the universe is filled, and actually dominated, by absolute, objective, universal truths gives the lie to the extreme relativism that prevails in many parts of this planet today. I do not believe postmodernism would survive an encounter with mathematics. A heightened and more widely spread understanding of mathematics is therefore likely to promote

[2] *Making mathematics count: the report of Sir Adrian Smith's inquiry into post-14 mathematics education* (937764), The Stationery Office, February 2004.

[3] 'Is mathematics an act of creation or a discovery? Many mathematicians fluctuate between feeling they are being creative and a sense they are discovering absolute scientific truths. Mathematical ideas can often appear very personal and dependent on the creative mind that conceived them. Yet that is balanced by the belief that its logical character means that every mathematician is living in the same mathematical world that is full of immutable truths. These truths are simply waiting to be unearthed, and no amount of creative thinking will undermine their existence.' (du Sautoy, *The Music of the Primes*, pp.33-4.)

cohesion in our society, for the very reason that it will cause individuals to look afresh at their own personal philosophy from the basis of a shared intellectual foundation.

Mathematics is also, paradoxically, about values. For reasons that they are sometimes embarrassingly unable to supply, mathematicians are permanently in search of beauty. As the English mathematician G.H. Hardy (1877-1947) put it: 'Beauty is the first test: there is no permanent place in this world for ugly mathematics.'[4]

Moreover, as the French mathematics professor Alain Connes put it, mathematics, 'is unquestionably the only *universal* language'[5] - the only one with which we could communicate with an alien intelligence from another planet or solar system.

*   *   *   *   *

We have therefore every incentive to break into the downward spiral and develop a national interest in mathematics. I propose as solutions:

(1) To equip teachers of mathematics with a greater sense of the *shape* and *structure* of their subject.
(2) To provide an historical framework for existing secondary school syllabuses.
(3) To introduce a new 'A' level in the History of Mathematics.
(4) To make mathematics accessible to the population at large.

(1) TEACHER TRAINING

The terror of mathematics begins in the classroom. Some GCSE textbooks in mathematics appear to have been written by individuals who have neither any real grasp of the subject themselves nor any gift for teaching others. Even intelligent pupils and adults have to *decipher* what is being taught before they can understand it and make it their own. 'Relevance' - not to mention social considerations - has been promoted above clarity and mathematical content. The concept of *proof* - getting the student to prove things - seems altogether lacking. One would never conclude from reading them that their goal was to entrance and delight with the kaleidoscope that is their subject, or to equip their students with resources that will be of the highest benefit to them for the whole of their future lives. However from today's inadequately taught school-leavers come tomorrow's ill-prepared mathematics teachers and writers of texts.

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[4] G.H. Hardy, *A Mathematician's Apology* (1940). 1992 Edition (Cambridge: CUP), p.85.

[5] Jean-Pierre Changeux and Alain Connes, *Conversations on Mind, Matter and Mathematics*, ed./tr. M.B. DeBevoise (Princeton, NJ: Princeton University Press, 1995), 10.

The cry that one hears repeatedly, that 'I could never do maths at school; couldn't understand what it was all about; and gave it up as soon as possible', is the most damning indictment. It seems to be commonplace to fire topic after topic at unfortunate pupils, with no connecting rhyme or reason explaining *why* these topics are important and *why they are being taught in this order*. This can only mean that the teachers themselves do not understand the *shape* and *structure* of their own subject, a charge that might justly be levelled at the authors of the GCSE textbooks in question.

So a recent Ofsted report into mathematical education found that

> The quality of teaching was the key factor in influencing students' achievement.[6]

In particular, high achievement in mathematics in the 14-19 year age group was found to be hindered by

> Teaching which presents mathematics as a collection of arbitrary rules and procedures, allied to a narrow range of learning activities in lessons which do not engage students in real mathematical thinking,[7]

and by

> Insufficient subject expertise amongst some teachers of GCSE mathematics and numeracy programmes; a lack of imagination and the confidence to try new approaches amongst A level teachers.[8]

By contrast, competence in mathematics was promoted most significantly by

> Secure subject knowledge on the part of the teacher, underpinning an approach to mathematics in which all topics are seen as part of a coherent set of related ideas, with clear progression and links to previous and future learning,[9]

Fundamental to the solution therefore is *to re-educate existing mathematics teachers, and to educate new ones, in the* structure *of their subject*. And on this point BRAINWAVES must declare an interest, having over the last eighteen years authored a textbook, one of whose intended functions is to support just such an exercise.

The book, now available from Section III of www.brainwaves.co.uk, is called *e, i & π: A Mathematical Drama in Three Acts*, and it sets out particularly to emphasise the logical connections between the

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6 *Evaluating mathematics provision for 14-19-year-olds* (Ref: HMI 2611), Ofsted, May 2006, p.1.
7 Ibid. p.2.
8 Ibid. p.3.
9 Ibid. p.2.

various elements taught. These are illustrated in the appended 'Baseline Flowchart', which shows how the various themes develop, interweave and cross-fertilise. So for instance Pascal's Triangle leads directly into the binomial theorem, which in turn feeds into the differential calculus. The binomial theorem, in its simpler and more complex manifestations, forms the linchpin between Act 1 (basics) and (Act 3) (series), and a spreadsheet is provided on disk for exploring this. (Act 2 comprises the calculus.) In its broad sweep the book seeks to show how the three numbers *e*, *i* and π (plus, in particular, Pascal's Triangle) grow together to be ultimately unified in Euler's celebrated identity

$$e^{ix} \equiv \cos x + i \sin x$$

From this follows what is generally considered to be the most beautiful equation in all mathematics,

$$e^{i\pi} = -1 \qquad \text{(or,} \quad e^{i\pi} + 1 = 0\text{),}$$

which is as startling and rock-like in its own way as the synthetic a priori proclamation

'I AM WHO I AM'[10]

is to the Old Testament. People have been converted to mathematics merely by discovering it.

This drama has then become the focus to which all manner of other topics have been interwoven, as summarised schematically in the Baseline Flowchart. Regular mention is made of the authors and dates of discoveries, providing human interest and historical continuity to support the logical development. Issues in the philosophy of maths are also raised, and a substantial glossary of reference material is provided at the end. The Bibliography from the book is also appended to this paper, slightly adapted, to which references are made below.

(2) GCSE AND 'A' LEVEL SYLLABUSES: AN HISTORICAL FRAMEWORK

The present writer is sceptical of school syllabuses which are constructed primarily as an introduction to 'modern' mathematics (in this context, paradoxically, of the last 150-200 years - a very short timescale in fact, given the extreme antiquity of the subject), laying for instance a strong early emphasis on sets, groups and other such structures. These often leave quite baffled pupils who do not continue to the higher levels where the justification for such topics becomes apparent. This seems to me to be like starting the teaching of history with the Napoleonic Wars. All too often they leave unanswered the question of what led mathematicians to posit such things in the first place. (So, Galois discovered group theory in 1829 because he had come to appreciate the impossibility of

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[10] Exodus 3:14.

solving the general quintic equation by the traditional method of extracting roots. But the question as to why was he investigating these is often left unasked.)

Far preferable, I suggest, would be an approach *which teaches mathematics very broadly according to the logical and historical sequences in which humanity discovered it*. This would give the enquirer an immediate answer to the question, why are we studying this topic? Each new topic as far as possible would be founded upon its historical predecessors. Students could then see the threads of themes that wove and interwove as the human race progressed along its mathematical pilgrimage. The narrative context would thus provide a perspective, hooks upon which each new technique or theorem may be hung. The question, why?, would then always have an answer.

We might follow the acquisition of fundamental numeracy skills by asking, what were the problems with which early civilisations were wrestling? This would lead on to an account of Euclidean geometry, which held sway for some two thousand years. Euclid after all, perhaps more than anything, offers abundant early opportunity to practise the art of proof. In due course would follow in order such milestones as Descartes' invention of coordinate geometry (1637), Pascal's Triangle (1653), the general binomial theorem (Newton, 1665), the joint discovery of the calculus by Newton and Leibniz (late seventeenth century), de Moivre's theorem (*c*.1707) and the unification achieved by Euler in 1748. Into a loose framework such as this, all kinds of other material could be inserted where hindsight and teacher experience suggests it might be helpful, especially where the logical and historical sequences do not coincide. This is what *e, i &* π has attempted to do. Teaching how mathematicians thought in the past will then provide an excellent basis from which to learn how they think in the present. One benefit of this approach will be that, whenever pupils leave off the study of mathematics, they will at least have acquired a coherent whole, with a start and a finish, and hence some recognisable rhyme or reason underlying it.

I find this approach to the teaching of mathematics strongly endorsed by Freeman Dyson's Foreword to Julian Havil's fascinating book, *Gamma*, pp.xv-xvi. He contrasts three different approaches:

> The first approach was the 'boot-camp' method of drill and exercise that prepared students well for examinations but often did not enable them to develop a real understanding of mathematics. It mostly failed to encourage students to see the beauty and enjoyment to be gained from their subject....
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> The second approach...was called the 'New Math' and was a reaction to the dullness and shallowness of the old way of teaching. The New Math teaching was based on the idea that children should learn to understand modern mathematical concepts before they learned to solve practical problems, hence students would learn about sets and relations before they had mastered multiplication and division. Students learned the vocabulary of modern mathematics without understanding the substance. After a few years of New Math, mathematical literacy declined precipitously....
>
> The third way is to use a historical approach to mathematics, teaching the practical skills that students need, but in the context of the history of the time when these skills were first developed.

This is precisely the concept which underlies *e, i & π.*

I would stress, however, that *e, i and π* is only a start and makes no claim to be comprehensive. It offers no applied mathematics, and omits a number of topics, such as differential equations, which might be desirable in an 'A' level syllabus but fall way outside its ambit.

(3) AN 'A' LEVEL IN THE HISTORY OF MATHEMATICS

The third proposal arises from the writer's experience of studying in 1995 the excellent course MA 290 in the History of Mathematics offered by the Open University. *Alongside the standard 'A' level courses in mathematics, schools and colleges should offer a parallel course in the History of Mathematics.*

For all its universality, mathematics is also a very human discipline. Its history is the history and common property of the human race. Its truths were learned by actual people pursuing genuine enquiries. Some found solutions themselves. Some raised questions they were unable to solve, passing them on to later generations. They came from many different races. There were Greeks like Pythagoras, Euclid and Archimedes, whose names today are household words. There were Chinese, who like the Persians discovered Pascal's Triangle long before Pascal. There were Arabs, who gave us *algebra* and arabic numerals, besides preserving for us the great mathematical heritage of the Greeks. There were Hindus, to whom we owe the digit zero which makes possible the positional notation that we use today. Later on there have been Britons like Newton and Hardy, and proliferations of French, Germans, Swiss, Italians and other Europeans. All of these make up one long and fascinating story.

To teach the history of mathematics and the great minds who discovered it will, I believe, put life back into a subject which was rightly dubbed by Gauss to be the 'Queen of the Sciences', but which is all too often perceived today to be dull, arid and dreary. It will attract students who might otherwise have turned elsewhere, and instil in them a passion for their subject. It will raise the levels of comprehension of those whom it attracts. And it will enable them to bridge in an unique way the divide between the arts and the sciences.

For a pilot scheme a possible basic textbook already exists in Hogben, *Mathematics for the Million*. If this is currently out of print doubtless demand will revive it.

Teachers would be equipped to teach the new 'A' level by a course such as that from the Open University referred to above, if still available, or equivalent. Their supporting textbooks might include Boyer and Merzbach, *A History of Mathematics*, and Fauvel and Gray, *The History of Mathematics: A Reader*.

(4) REACHING THE WIDER POPULATION

The transformation we are hoping to achieve will need to be shared with the wider population who are already beyond school or university age. Many parents, for instance, will want to be able to keep abreast of what their children are learning, if only to assist them. Others will be surprised to discover that mathematics can be a source of delight and inspiration once they can lose their fear of it. There are aspects of mathematics which can enrich, amuse and entrance anyone. These will include:

History, biography (see Bibliography)

Recreational mathematics, including:

Brain-teasers and puzzles
Patterns
Games
Curiosities
Codes
Competitions

Home computing (e.g. distributed computing, such as the Great Internet Mersenne Primes Search (GIMPS), a networking of home computers which performs 2 trillion calculations per second continuously; and ZetaGrid, which currently employs over 11,000 workstations to compute 1 billion zeros for the Riemann zeta function every day.[11])

Mathematical fashions come and go. The Fibonacci sequence, 1, 1, 2, 3, 5, 8, 13,..., first discovered by Leonardo of Pisa (*c.*1175-1250), has an enduring popularity and is now something of an industry of its own with at least one website (www.goldennumber.net) and a journal, *The Fibonacci Quarterly*, dedicated to it. Today, on the popular bookshelves it is very much the *prime numbers* which are making the running (cf. du Sautoy, Wells, *Prime Numbers* and Hoffman).

How are we to set about our endeavour? If we begin as if we were trying to propagate a new political philosophy or creed which we passionately believed was an unqualified universal good, we shall not be wide of the mark.

The parallel is not unjust. What we are talking about is a thoroughly wholesome delight, open to all, but of whose reality few who have not experienced it will ever have guessed. For evidence of this we

[11] See respectively Wells, *Prime Numbers*, pp.115, 153-5 and pp.28, 47, 208)

need look no further than Martin Gardner's excellent series of books on recreational mathematics (see Bibliography). Our contention is that, given imaginative, dedicated and multi-faceted leadership from the top, such a prospect is indeed realisable for a large fraction of the population, and that the benefits of it, direct and indirect, will be universal.

All the media offer possibilities. For instance

Television: The Open University has been teaching mathematics on traditional television channels for decades. Digital and cable TV offer wider possibilities, such as a Mathematics Channel on the same lines as the History Channel. There is scope for programmes as imaginative as the brilliant BBC *Master Game* series which popularised chess in the 1980s. One example would be Simon Singh's outstanding BBC Horizon documentary on *Fermat's Last Theorem* in 1996, greatly assisted by the power of modern computer graphics. Another is Marcus du Sautoy's excellent recent series on the history of mathematics. Older viewers particularly may welcome the refreshing and constructive alternative to the present daily diet of sex, swearing, hype, violence, soap, quiz games and confrontation.

Internet: This is already awash with mathematical websites - see for instance the bibliography to Wells, *Prime Numbers.* I understand that there are now university courses available online.

Newspapers and magazines: There is already a long tradition of puzzles and brain-teasers, and in particular the present Su Doku phenomenon, which offers powerful evidence of a market for self-education in the art of thinking. To these could be added regular features on topics of mathematical interest, in the tradition of Martin Gardner's world-renowned articles on recreational mathematics in *Scientific American* (reproduced in paperbacks such as those just referred to).

Books: Again see Bibliography. Note particularly Sawyer, *Mathematician's Delight*, which was expressly written to dispel the fear of mathematics. Also Crilly, *50 Mathematical Ideas You Really Need to Know:* there must be many who would find their horizons immeasurably broadened and their minds excited by reading this.

It is to be hoped that various different types of institution would want to contribute as the campaign progressed. Sponsors and supporters might include

Universities
Industry
Mathematical organisations

A host of mathematical journals.

Support and technical leadership may also be expected from writers who are already in the business of popularising mathematics - again, see Bibliography.

Benefits might be expected, for instance, within prisons, at a time when reoffending is at an all-time peak. It has already been found how prisoners can benefit from exposure to the world of Shakespeare. Much the same, I expect, would be true if, within their confines, more were able to enter into and explore the unfettered world of mathematics than current educational facilities can offer. There may be scope, too, for integrating disaffected Muslim youth by awakening them to the tremendous contribution to mathematics made by their Islamic forebears. Possibilities for life-enrichment of this kind could doubtless be imagined in many other spheres.

Anyone who doubts the potential effectiveness of off-beat, single-issue campaigns such as this will perhaps be convinced by Jamie Oliver's brilliantly orchestrated crusade to improve the quality of food in our schools, culminating in a change in the law in 2006. Where there's a will....

CONCLUSION

I propose therefore that a concerted effort be made to raise the mathematical consciousness of the nation by enabling people of all ages to rediscover the delight, the excitement, the discipline and the fascination of mathematics. This is not only achievable. It is achievable at an affordable cost. It offers innumerable benefits to individuals, to the economy and to society generally, for the price of no obvious disadvantages. Every citizen who learns to Think is a guarantor of our national freedoms. Even fragmentary success will therefore be a boon. There is everything to play for.

'Mathematics is the gymnasium of the mind.'

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B2: RECREATIONAL

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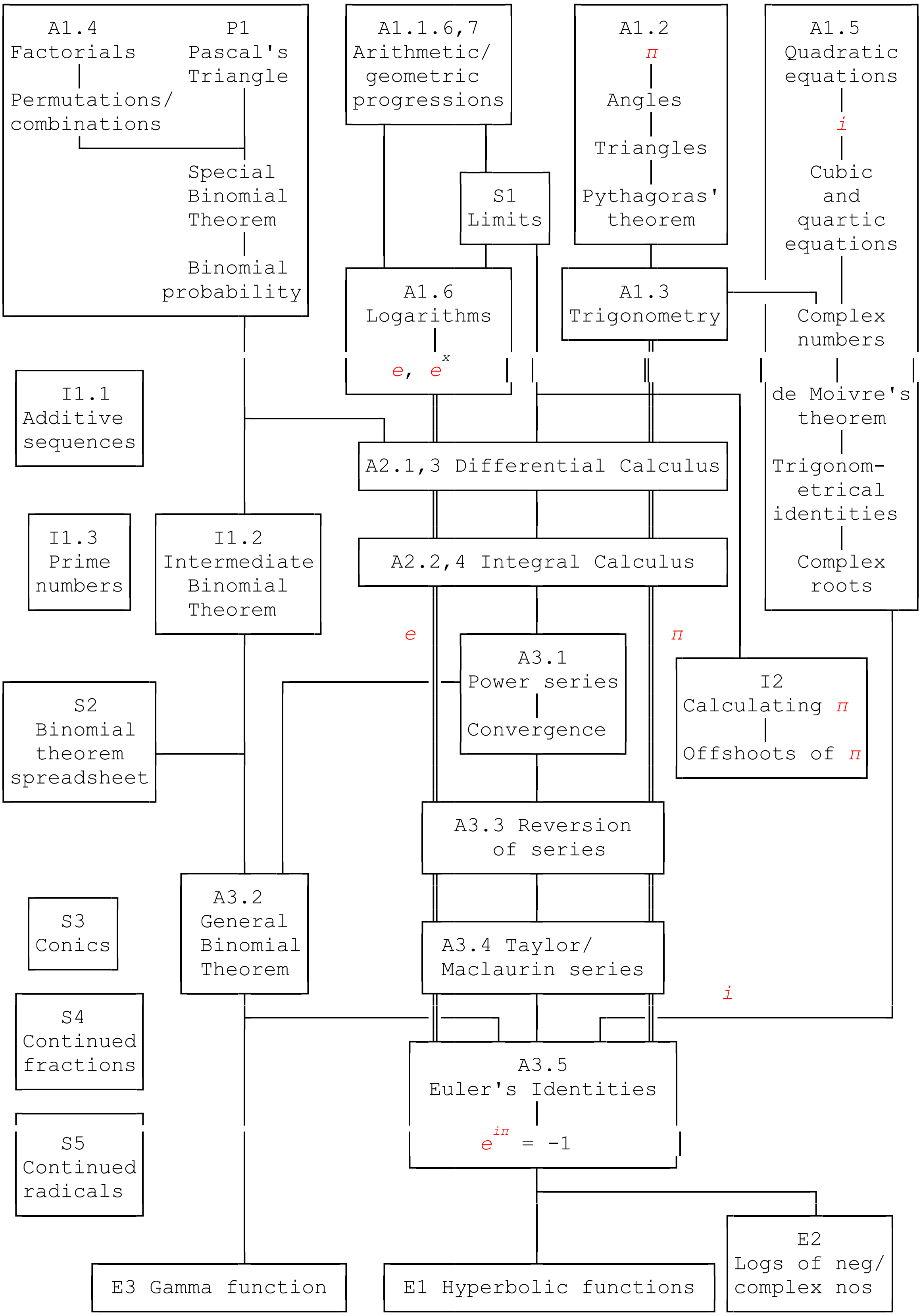

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