BRAINWAVES REPORT BW/031: WHY PI IS AN IMPOSTOR[1]

In affectionate memory of Col. D.H.W. 'Skinner' Sanders, OBE, RM.

The mathematician's patterns, like the painter's or the poet's, must be beautiful*; the ideas, like the colours or the words, must fit together in a harmonious way. Beauty is the first test: there is no permanent place in the world for ugly mathematics.*[2]

THE DEFINITION OF π

π is traditionally accorded the highest honours in mathematics on account of the wide variety of contexts in which it is crucial. How do we define it? Consult any textbook, dictionary of mathematics or website, and you will almost certainly find a definition that relates π to circles. π, you will read in so many words, is the ratio of the circumference C of a circle to its diameter d, given approximately by 3.14. And so it has been known, by various different names and to varying degrees of precision, throughout history. Nevertheless I find three objections to all proposed definitions of π which rely on the proportions of circles.

(1) Definitions in terms of circles presuppose a prior knowledge of geometry such as the definition of the circle itself. But defining geometry has proved more difficult than was conceived, for instance, by Euclid and his followers. Rigour is not to be obtained as easily as we had hoped.

(2) As many mathematicians will concede, the formulae for the circumference and the area of a circle illustrate *properties* of π – what it *does* – where what we need is a *definition* of what it *is*. π is a *number*. If we define it as a fact about geometry it will carry a baggage that is totally out of place in the many mathematical contexts where geometry is wholly inapplicable.

(3) Definitions of π in terms of circular measure give us no immediate means of calculating it. That is why the textbook definitions always supply us with a very approximate value. However many decimal places they give, they offer no rule for obtaining the rest. *This geometrical definition does not actually define the quantity in question!* Given a circle and a measuring device, how according to the standard definition am I to compute the hundredth, thousandth or millionth decimal place? I can't!

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[1] This paper forms a Postscript to Martin Mosse, '*e*, *i* & π' (2013) (which may be downloaded from the Books section of my website www.brainwaves.org.uk). I would like to acknowledge the valuable comments and criticisms on earlier drafts of this paper received from Nigel Rockliffe, who like myself had the pleasure of being taught mathematics at an early age by Col. Sanders; also some incisive comments from Nick Salkilld.

[2] G. H. Hardy, *A Mathematician's Apology* (1940; Cambridge: Cambridge University Press, 1992), 85.

The history of mathematics is decorated with numerous methods for calculating or approximating π. The story has been told many times. Archimedes famously provided the first, by squeezing a circle between regular polygons just outside and just inside it, thereby providing a bracket within which π must lie. Different civilisations came up with different approximations. In my book '*e*, *i* & π'[3] I have provided a historical survey of how different mathematicians down the ages found formulae of varying complexity from which π may be calculated to any required precision. The first theoretically precise expression for π known to the West was discovered by the French mathematician Viète in 1593, as

$$\pi = \frac{2}{\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}+\tfrac{1}{2}\sqrt{\tfrac{1}{2}+\ldots}}}}}} \tag{1}$$

In 1655 came Wallis's product formula,

$$\frac{\pi}{4} = \frac{2}{3}\times\frac{4}{3}\times\frac{4}{5}\times\frac{6}{5}\times\frac{6}{7}\times\ldots, \tag{2}$$

the first requiring only algebraic operations, rather than the square roots employed by Archimedes and Viète. From here Newton in 1666, with the benefit of the integral calculus he had discovered, gave the somewhat daunting and ungainly expression

$$\frac{\pi}{6} = \frac{1}{2}+\frac{1}{2}\left(\frac{1}{3\times 2^3}\right)+\frac{1\times 3}{2\times 4}\left(\frac{1}{5\times 2^5}\right)+\frac{1\times 3\times 5}{2\times 4\times 6}\left(\frac{1}{7\times 2^7}\right)+\ldots \tag{3}$$

Finally, by contrast, came the beautiful and elegantly simple

$$\frac{\pi}{4} = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\ldots \tag{4}$$

named Leibniz' series after Gottfried Leibniz, who in 1674 gave it as the area of a quarter circle of unit radius, although Gregory had reached it three years earlier and it was known to the Indian mathematician Madhava three centuries before that. This I contend lays claim to be the most fundamental definition of π known to mathematics. It suffers from none of the disadvantages inherent in the circular definitions discussed above.

(1) For although its derivation lay in geometry, in this definition no geometrical assumptions are made whatever. It has shed the geometrical origins from which we first intuited its existence.

3 Martin Mosse, op. cit., Interlude I2.1; see also Section A1.2.1.

(2) π is at last defined by what it *is* – a very special but simply defined *number* – rather than what it *does*.

(3) π is revealed as a simple sum of an infinite series of terms whose magnitudes are just the reciprocals of the odd numbers. It requires no more than the basic operations of arithmetic. A child could understand it. An intelligent child could attempt to compute the first few digits (although convergence is pitifully slow as can be seen from Figure 1 below; but that is no objection in a definition). Anyone with a computing aid can in principle calculate π from this definition to a theoretically unlimited degree of precision.

On all these grounds I submit that Leibniz' series presents a very much stronger case than the circular formulae for defining π. This means we are free to use it in any branch of mathematics without the geometrical overtones which it has discarded.

THE RIGHT CONSTANT

However, we have skated over one point. For although Leibniz' series provides us with the optimal method of defining π, it is actually $\pi/4$ rather than π that is the limiting sum of the series! And now that we have left the circular context in which we started, we are free to ask the question: What constant do we really want to define? And this is a relevant question today. For π has its rivals. Most notably the American mathematician Michael Hartl maintains that the 'circular constant' should really be defined not as *C*/*d* but as *C*/r, denoted τ (tau), where *r* is the radius. You can find his case persuasively put as 'The Tau Manifesto' on the internet.[4] Thus $\tau = 2\pi$. It seems that we are free to choose whichever we prefer. But for all his logic he does not escape the three fundamental objections given above to all attempted definitions which rely upon geometrical ratios about circles. I have found no indication in his paper of how he proposes to calculate τ and no mention of series.

Continuing to scan down my survey of the methods of calculating π,[5] it is striking how many of the formulae given there, like Leibniz' series, compute not π but $\pi/4$. Besides (2) and (4) above we have

$$\pi/4 = \tan^{-1}(1/2) + \tan^{-1}(1/3) \qquad (5)$$

and Lord Brouncker's beautiful continued fraction for the inverse

[4] Michael Hartl, 'The Tau Manifesto' (2013), http://www.tauday.com/tau-manifesto.pdf.
[5] Mosse, op. cit., Interlude I2.1, 'Calculating π '.

$$\frac{4}{\pi} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{2 + \dots}}}} \qquad (6)$$

Then Machin's formula (1706) gave

$$\pi/4 = 4 \tan^{-1}(1/5) - \tan^{-1}(1/239) \qquad (7)$$

one of many more arctangent formulae such as Euler's (*c*.1755),

$$\pi/4 = 5 \tan^{-1}(1/7) + 2 \tan^{-1}(3/79) \qquad (8)$$

After which in 1844 came, with three terms,

$$\pi/4 = \tan^{-1}(1/2) + 2 \tan^{-1}(1/5) + \tan^{-1}(1/8) \qquad (9)$$

used by Dase to compute 200 digits, followed in 1893 by Loney's

$$\pi/4 = 3 \tan^{-1}(1/4) + \tan^{-1}(1/20) + \tan^{-1}(1/1985) \qquad (10)$$

from which William Shanks correctly computed 527 digits. Then in 1896 Störmer derived

$$\pi/4 = 6 \tan^{-1}(1/8) + 2 \tan^{-1}(1/57) + \tan^{-1}(1/239) \qquad (11)$$

which in 1961, early in the computer age, was used by Shanks (Daniel, no relative of William above) and Wrench to obtain 100265 decimal places.

Take another look at (1) above. Why the extraneous 2 as the numerator? It looks as though it might be more felicitously rewritten as a straight reciprocal

$$\frac{\pi}{4} = \frac{1}{2\sqrt{\tfrac{1}{2} + \tfrac{1}{2}\sqrt{\tfrac{1}{2} + \tfrac{1}{2}\sqrt{\tfrac{1}{2} + \tfrac{1}{2}\sqrt{\tfrac{1}{2} + \tfrac{1}{2}\sqrt{\tfrac{1}{2} + \dots}}}}}} \qquad (12)$$

Note how anyone computing these expressions, after calculating the value of the right hand side, always has to end by multiplying the limit by 4. That 4 is *ugly*. What is it doing there? It is as if mathematics itself is trying to tell us something. It is now plain that the universal constant we want is the limiting value of Leibniz' series (4), which we define as

$$\psi = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + ... = 0.785398163397448309615660 8... \qquad (13)$$

This may alternatively be represented by its simple continued fraction [0; 1, 3, 1, 1, 1, 15, 2, 72,…], or by the common fraction 355/452 to six significant figures.

And so we unveil ψ (psi), one of the essential and most fundamental constants in the whole of mathematics, long eclipsed by the very much more visible impostor π, so readily detected from the universally available circle. ψ was not recognised for what it is on account of the unchallenged prevalence of the tempting but erroneous geometrical definitions of π discussed above. Only when series were first explored in the seventeenth century did it become possible to recognise the significance of the true claimant ψ as underlying π.

Now we see why multiples and powers of π crop up so frequently in mathematics. They owe their import to the fact that they are all multiples or powers of ψ, which is the limit of the basic and most fundamental form of the series (13) in question, and numerous other series besides. They have little if any real significance of their own. So $\pi = 4\psi, \tau = 8\psi$. Different areas of mathematics appear to work best with different manifestations of ψ.

Consider the sister series for the natural logarithm of 2:

$$\ln 2 = 1 - 1/2 + 1/3 - 1/4 + 1/5 - ... = 0.69314718... \qquad (14)$$

Here again it is the sum (limit) of the series which yields the required mathematical constant, and not a multiple of it.

We may compare also that equally central constant *e*, which first made its appearance in the computation of compound interest and logarithms in the seventeenth century,[6] when it was found that

$$e = \lim_{n \to \infty} (1 + 1/n)^n \qquad (15)$$

From here manipulation gave

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + ... \qquad (16)$$

which, quite correctly, is the formal definition of *e*. This requires no extraneous mathematics such as exponentiation beyond the basic algebraic operations. It tells us what the number *e*

6 See the account in Mosse, op. cit., Section A1.6.2, '*e*'.

is, not what it *does*. Its multiple applications in mathematics are totally independent of the context in which it first appeared, making no reference to interest computations. So it is with ψ and circles.

Thus ψ and *e* together, defined as they are in terms of simple relationships between the integers, illustrate Kronecker's celebrated dictum, 'God made the integers; all the rest is the work of man.' It also explains the mystifying nature of both transcendental constants, whose seemingly random sequence of digits appears at first sight to be totally arbitrary, plucked out of the air, and devoid of pattern or regularity; yet they are of immeasurable importance to the whole of mathematics, and indeed to the whole universe.[7] They are merely the sums of simple manipulations of the integers in formulae that in both cases exhibit a striking regularity and beauty. As such they illustrate wonderfully the quotation from Hardy which heads this essay.

Martin Mosse,
BRAINWAVES,
January 2015.

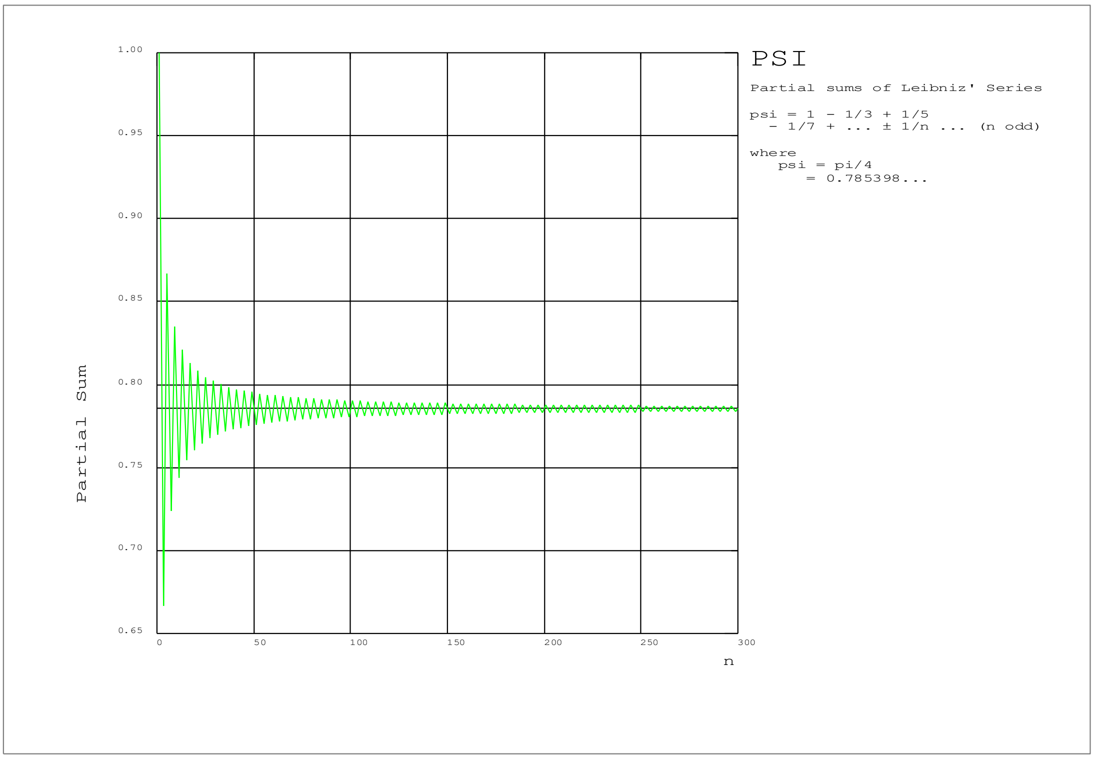

Figure 1: Convergence of Leibniz' Series for ψ.

[7] See the comment reported by W.W Rouse Ball and H.S.M. Coxeter on *e* and π:

> 'I recollect a distinguished professor explaining how different would be the ordinary life of a race of beings for whom the fundamental processes of arithmetic, algebra and geometry were different from those which seem to us so evident; but, he added, it is impossible to conceive of a universe in which *e* and π should not exist.'

(*Mathematical Recreations & Essays*, Twelfth Edition (Toronto: University of Toronto Press, 1974)), p.348.

APPENDIX

THE CIRCUMFERENCE AND AREA OF A CIRCLE

When π was defined in terms of the circumference $C = 2\pi r$ and area $A = \pi r^2$ of a circle radius *r*, no need was seen to prove these two expressions, precisely because they were definitions. The problem became one of establishing the value of π. This way of doing things taught us very little about arcs, angles and radii, and their interrelationships. Now that we have chosen to begin with a numerical definition of ψ (and hence of π), it is incumbent on us to establish formulae for these two features of the circle, based on the known value of ψ. To do this, we need first to understand the nature of angles.

DEFINITIONS OF AN ANGLE

We offer two definitions of an angle.

(A) In radians. The angle θ radians subtended at the centre of a circle of radius *r* by an arc of length *s* is given by $\theta = s/r$, from which

$$s = r\,\theta. \qquad (17)$$

(B) In degrees. 1 degree (1°) is the angle subtended at the centre of a circle by an arc whose length is 1/360 of the circumference. Thus 360° represents a full circle.

CIRCUMFERENCE OF A CIRCLE

Starting now with the result, already known in the seventeenth century,[8]

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \qquad |x| \le 1 \qquad (18)$$

rewritten as

$$\tan^{-1} x = \int \frac{dx}{1+x^2} \qquad |x| \le 1 \qquad (19)$$

we expand the RHS by polynomial long division to get

$$\tan^{-1} x = \int 1 - x^2 + x^4 - x^6 - \ldots\, dx \qquad |x| \le 1 \qquad (20)$$

which can be integrated to obtain Gregory's series (1671):

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots, \qquad |x| \le 1 \qquad (21)$$

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[8] Proved as A2.3.2 (7) on p.210 of '*e*, *i* & π'.

Putting $x = 1$, and comparing with Leibniz' series

$$\psi = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots, \qquad \text{(13) above}$$

it is evident that $\tan^{-1} 1 = \psi$. That is, ψ is the (principal value) angle in radians whose tangent is 1. This may be visualised in the isosceles right-angled triangle depicted in '*e*, *i* & π', A1.3.3 Figure 1, from which it is plain that ψ radians = 45° (half a right angle), since tan 45° = 1.

But from definition (B) of an angle above, 45° = 45/360 = 1/8 of a circle. So a whole circle embraces 8ψ radians. Hence from equation (17) above, the circumference C of a circle radius r is given by

$$C = 8\psi r. \qquad (22)$$

AREA OF A CIRCLE

A proof of the traditional formula for the area of a circle, given that for its circumference, is to be found on p.200 of '*e*, *i* & π', where it is used to illustrate the basic concept of integration. We may think of it conceptually in brief as follows.

The area A of a circle will increase, as its radius r increases, at a rate proportional to the length of its circumference, which is itself a constant multiple 8ψ of r (equation (22)). The larger the circumference, the greater will be the increase in A for a given increase in r. That is,

$$\frac{dA}{dr} = 8\psi r \qquad (23)$$

Simple integration with respect to r yields

$$A = 4\psi r^2 \qquad (24)$$

which is the formula for the area of a circle.

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I suggest that there is much more of value to be learned about mathematics from *proving* these two formulae than there is by merely defining them to be true!